

Second Edition

S**olutions**
Test Drill I
Test Drill II

Control Systems Drill

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Solutions

Control System

Test Drill I

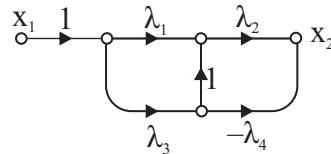
Q1. Apply Mason's gain rule to the signal flow graph equivalent to block diagram as sketched below, to get

$$\frac{x_2}{x_1} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}; P_1 = \lambda_1 \lambda_2, P_2 = \lambda_2 \lambda_3, \Delta_1 = \Delta_2 = 1$$

$$\text{and } \Delta = 1 - (-\lambda_2 \lambda_4) = 1 + \lambda_2 \lambda_4$$

$$= \frac{(\lambda_1 + \lambda_3) \lambda_2}{1 + \lambda_2 \lambda_4} = \frac{(10 + 5) \lambda_2}{1 + 0.5 \lambda_2}$$

$$x_2 = 18x_1 \Rightarrow \frac{15\lambda_2}{1 + 0.5\lambda_2} = 18 \Rightarrow \lambda_2 = 3$$



Q2. $G_1|_{x_2=0} = \frac{y}{x_1} = \frac{P_i \Delta_i}{\Delta}; i = 1, 2, 3, 4 \text{ and } \Delta_i = \Delta = 1$

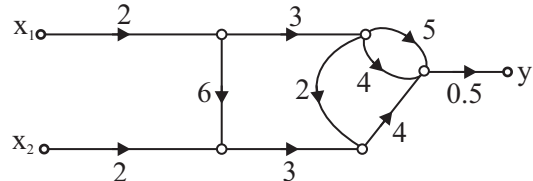
$$= 0.5 [2 \times 3 \times 4 + 2 \times 3 \times 5 + 2 \times 3 \times 2 \times 4 + 2 \times 6 \times 3 \times 4]$$

$$= 123$$

$$G_2|_{x_1=0} = \frac{y}{x_2} = \frac{P_1 \Delta_1}{\Delta}; P_1 = 2 \times 3 \times 4 \times 0.5 = 12, \Delta_1 = \Delta = 1$$

$$= 12$$

$$M = \frac{G_1}{G_2} = \frac{123}{12} = 10.25$$



Q3. $Y(s) = G(s) \times R(s) = \frac{G(s)}{s} = \frac{1}{s} + \frac{0.2}{(s+60)} - \frac{1.2}{s+10} = \frac{600}{s(s+10)(s+60)}$

$$G(s) = \frac{600}{s^2 + 70s + 600} \text{ and } \omega_n = \sqrt{600} \text{ and damping ratio } \xi = \frac{70}{2 \times \sqrt{600}} = 1.43$$

Q4(d) Loop function, $G(s) = \frac{\lambda}{s} A(s) = \frac{\lambda(2s+1)}{s(5s+1)(s+2)^2}$; *hint*: this is type 1 system

$$K_p = \lim_{s \rightarrow 0} G(s) = \infty \text{ and } K_v = \lim_{s \rightarrow 0} s G(s) = \frac{\lambda}{4}$$

$$e(\infty) = \frac{1}{1 + K_p} + \frac{4}{K_v} = 0 + \frac{16}{\lambda} \text{ and } e(\infty) < 0.1 \Rightarrow \frac{16}{\lambda} < 0.1 \Rightarrow \lambda > 160$$

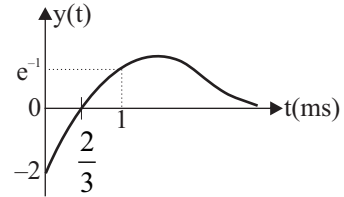
$$\text{Q5(a)} \quad Y(s)|_{X(s)=1} = \frac{1-2s}{(s+1)^2} \equiv \frac{a}{(s+1)^2} + \frac{b}{s+1}$$

$$a = 1 - 2s|_{s=-1} = 3 \quad \text{and} \quad b = \frac{d}{ds}(1-2s)|_{s=-1} = -2$$

$$Y(s)|_{X(s)=1} = \frac{3}{(s+1)^2} - \frac{2}{s+1}$$

$$\text{Unit impulse response, } y(t) = 3te^{-t} - 2e^{-t} \\ = e^{-t}(3t - 2)$$

$$y\left(\frac{2}{3}\right) = 0, \quad y(\infty) = 0, \quad y(1) = e^{-1} \quad \text{and} \quad y(t) > 0, \quad \text{for } t > \frac{2}{3}$$



$$\text{Q6(b)} \quad 1 + \frac{K(s-1)}{s(s+2)} = 0 \Rightarrow s^2 + (2+K)s - K = 0$$

For stability $K < 0$ and $K > -2$, that is, $-2 < K < 0$

$$\text{Q7. } s^3 + 14s^2 + (45+k)s = 0 \text{ can be put in the form } 1 + \frac{k(s+1)}{s^3 + 14s^2 + 45s} = 0 \quad \text{or} \quad 1 + \frac{k(s+1)}{s(s+5)(s+9)} = 0$$

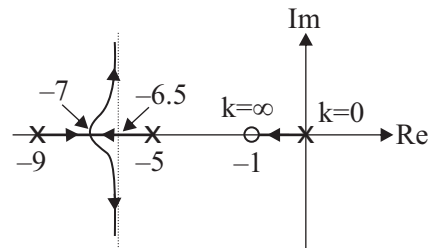
For sketching root locus, the product $GH = \frac{k(s+1)}{s(s+5)(s+9)}$; $0 \leq k \leq \infty$

$$\text{Centroid, } \sigma_A = \frac{-5 - 9 - (-1)}{3 - 1} = -6.5 \quad \text{gives} \quad \lambda = 6.5$$

$$\theta_A = \pm 90^\circ \quad \text{and} \quad k = -\frac{s^3 + 14s^2 + 45s}{s+1}$$

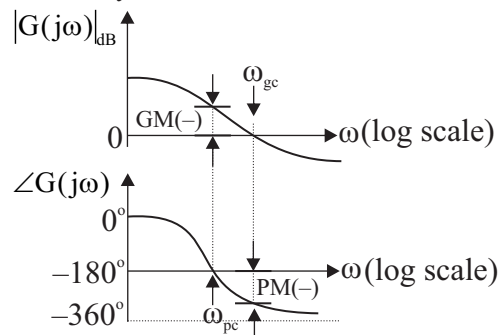
$$\frac{dk}{ds} = -\frac{d}{ds} \left[\frac{s^3 + 14s^2 + 45s}{s+1} \right] = 0$$

gives break away point at $s \cong -7$



Q8 (b) A cascade control system is beneficial only if the dynamics of inner loop is significantly faster than that of outer loop. Else, there arises a risk of interaction between the two loops leading to instability. A system with smaller time constant, exhibits faster dynamics.

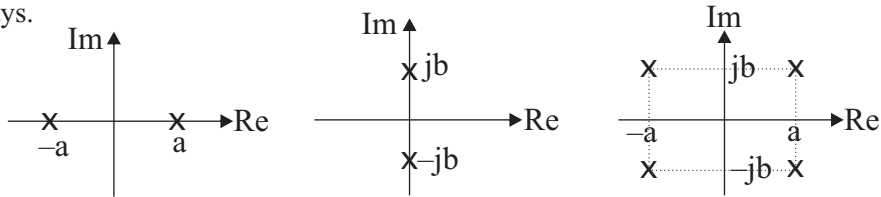
Q9 (c) If $\omega_{gc} > \omega_{pc}$, both GM (gain margin) and PM (phase margin) will be negative, and closed loop system will be unstable.



$$\text{Q10. } G(s) = \frac{75}{15\left(1 + \frac{s}{15}\right)(s^2 + 4s + 100)} \cong \frac{5}{s^2 + 4s + 100} = G_a(s) \Rightarrow K = 5$$

Note that $G(0)$ and $G_a(0)$ both are equal to 0.05. While approximating a high order system by a low order system, dc gains are matched in order to match their steady state responses.

Q11(c) The presence of entire row of zeros indicate presence of symmetrically located roots in either of following ways.



In routh array of a system of order seven, if s^3 row has all zero entries and there are two changes in sign below s^4 row, there will be 2 RHP roots and so only 2 LHP roots located symmetrically. There will be no root on $j\omega$ axis. The remaining 3 roots will be in the LHP.

Note that there are no changes in sign above s^4 row.

The Routh criterion provides actual location of IA roots, if any.

$$\text{Q12. } e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.05 \text{ give } \xi = 0.69$$

The characteristic equation, $s^2 + as + 16$ gives $\omega_n = 4$ and $\xi = \frac{a}{8}$

$$\frac{a}{8} = 0.69 \text{ gives } a = 5.52$$

$$\text{Q13. } K_V = \lim_{s \rightarrow 0} s \left[\frac{672(s+5)}{s(s+6)(s+7)(s+\lambda)} \right] = \frac{3360}{42\lambda}$$

Note $e(\infty) = \frac{R}{K_V}$ to input $r(t) = Rt$ $u(t)$

$$e(\infty) = \frac{5}{K_V} = \frac{5 \times 42\lambda}{3360} = 6.25 \times 10^{-2} \lambda$$

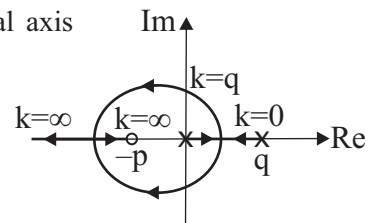
$$6.25 \times 10^{-2} \lambda = 10\% \text{ of } 5 = 0.5 \text{ gives } \lambda = \frac{0.5}{6.25 \times 10^{-2}} = 8$$

Q14(a) The characteristic equation is $s^2 - qs + ks - kp = 0$. For $k > 0$, p must be negative, that is, $p < 0$ for the system to be stable. Note that all coefficients must be positive.

If $q < 0$, $G(s) = \frac{k(s-p)}{s(s-q)}$ will have non origin pole on negative real axis

and system will be stable for all $k > 0$.
Since, the system is stable only for $k > q$,
the non origin pole of $G(s)$ must lie on
positive real axis, that is, $q > 0$.

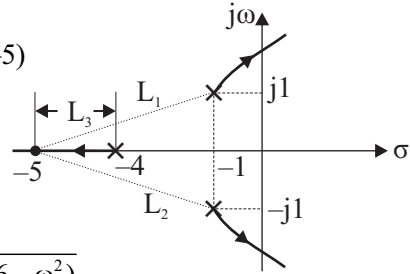
For $p < 0$ and $q > 0$, root locus is sketched for more insight.



Q15. $4k = \Pi(\text{phasor lengths from poles to the point at } s = -5)$

$$= L_1 \times L_2 \times L_3 = \sqrt{4^2 + 1^2} \times \sqrt{4^2 \times 1^2} \times 1 = 17$$

$$k = \frac{17}{4} = 4.25$$

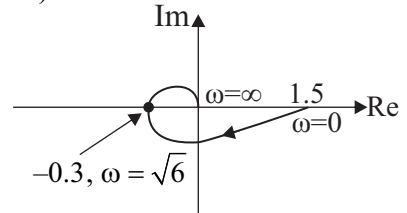


Q16(d) $G(s) = \frac{6}{s^3 + 4s^2 + 6s + 4}$ and $G(j\omega) = \frac{6}{(4 - 4\omega^2) + j\omega(6 - \omega^2)}$

$$\text{Im} [G(j\omega)] = 0 \text{ for } \omega = \sqrt{6}$$

$$\text{Re} [G(j\omega)] \Big|_{\omega=\sqrt{6}} = \frac{6}{4 - 4 \times 6} = -\frac{3}{10}$$

$$\text{GM} = 20 \log(10/3) = 10.46 \text{dB}$$



Q17. The transfer function $G(s) = \frac{1/Cs}{R + 1/Cs} = \frac{1}{1 + sRC} = \frac{1}{1 + s \times 10^6 \times 10^{-6}} = \frac{1}{s+1}$

$$G(j\omega) = \frac{1}{j\omega + 1}, |G(j\omega)| = \frac{1}{\sqrt{\omega^2 + 1}} \Big|_{\omega=1} = \frac{1}{\sqrt{2}} \text{ and } \angle G(j\omega) \Big|_{\omega=1} = -\tan^{-1}(1) = -45^\circ$$

$$\text{The output } y(t) = \frac{1}{\sqrt{2}} \sin(t + \phi - 45^\circ)$$

For $\phi = 45^\circ$, $y(t) = \frac{1}{\sqrt{2}} \sin t$ and there will be no transients.

Q18(b) The characteristic equation $|sI - A| = 0$ gives $\begin{vmatrix} s & 0 \\ 0 & s \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -100 & -20 \end{vmatrix} = 0$

$$\begin{vmatrix} s & -1 \\ 100 & s+20 \end{vmatrix} = 0 \Rightarrow s^2 + 20s + 100 = 0$$

$$\omega_n = 10 \text{ and } \xi = \frac{20}{2 \times 10} = 1$$

System will exhibit critically damped dynamics.

Q19. $G(s) = \frac{50}{s+50} = \frac{1}{1+(s/50)}$ has time constant $t_c = \frac{1}{50} = 0.02 \text{ sec} = 20 \text{ ms}$

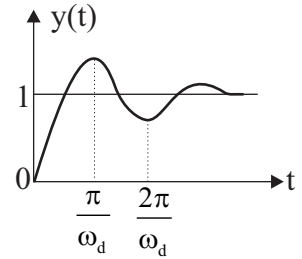
The settling time $t_s = 4t_c = 80 \text{ ms}$ and $|t_c - t_s| = 60 \text{ ms}$

Q20. $Y(s) = G(s)R(s) = \frac{5(s+2)}{s(s+5)} = \frac{2}{s} + \frac{3}{s+5}$

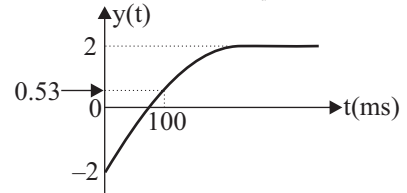
$y(t) = 2 + 3e^{-5t}$ gives forced component of response $\lambda=2$

and natural response at $t=0.2 \text{ second}$ is $\mu = 3e^{-5 \times 0.2} = 3e^{-1}$. Thus, $\frac{\lambda}{\mu} = \frac{2}{3e^{-1}} = 1.8$

Q21(b) The first undershoot time $= \frac{2\pi}{\omega_d}$; $\omega_d = 7$
 $= \frac{2\pi}{7} = 0.898 \text{ sec.}$



Q22 The response $Y(s) = \frac{10-s}{s+10} \times \frac{2}{s} = \frac{2}{s} - \frac{4}{s+10}$
 $y(t) = 2 - 4e^{-10t}$ and $y(t)|_{t=100\text{ms}} = 2 - e^{-10 \times 0.1}$
 $= 0.53$



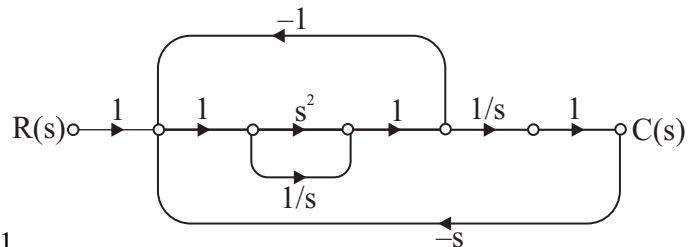
Q23(a) $P_1 = s^2 \times \frac{1}{s} = s$, $\Delta_1 = 1$

$P_2 = \frac{1}{s^2}$, $\Delta_2 = 1$

$\Delta = 1 - \left[-s^2 - \frac{1}{s} - \frac{s^3}{s} - \frac{s}{s^2} \right]$

$= 1 + 2s^2 + \frac{2}{s}$

$\frac{C(s)}{R(s)} = \frac{s + \frac{1}{s^2}}{1 + 2s^2 + \frac{2}{s}} = \frac{s^3 + 1}{s(2s^3 + s + 2)}$



Q24. $Y(s) = G(s)R(s)$; $R(s) = 1$, $r(t) = \delta(t)$

$G(s) = Y(s) = \frac{b}{(s+5)^2 + b^2} + \frac{s+5}{(s+5)^2 + b^2} = \frac{s+b+5}{(s+5)^2 + b^2}$

$G(0) = 0.2 \Rightarrow \frac{b+5}{25+b^2} = 0.2$ or $b = 0,5$

Since , b can not be 0, $b = 5$

Q25(d) $L(s) = \frac{k(z-s)}{s-p}$; $z > 0$ and $p > 0$

Characteristic equation , $1 + L(s) = 0 \Rightarrow (1-k)s + (kz-p) = 0$

For the system to be stable , either both coefficients $(1-k)$ and $(kz-p)$ be positive or both be negative , that is ,

either $(1-k) > 0$ and $(kz-p) > 0 \Rightarrow k < 1$ and $p < kz \Rightarrow \frac{p}{z} < k < 1$

or $(1-k) < 0$ and $(kz-p) < 0 \Rightarrow k > 1$ and $p > kz \Rightarrow 1 < k < \frac{p}{z}$

Solutions

Control System

Test Drill I

Q1d Characteristic equation: $1 + \frac{16(\lambda s + 1)}{s + (s + 0.8)} = 0$ or $s^2 + (0.8 + 16\lambda)s + 16 = 0$

$$\omega_n = 4 \text{ and } \xi = \frac{0.8 + 16\lambda}{2 \times 4} = 0.1 + 2\lambda ; \lambda \in [0, 0.8]$$

$$\text{For } \lambda \in [0.45, 0.8], \xi \in [1.0, 1.7]$$

and system will not exhibit sinusoidal dynamics.

In fact, the dynamics will be exponential; critically damped for $\xi = 0.45$ and overdamped for $0.45 < \xi \leq 0.8$.

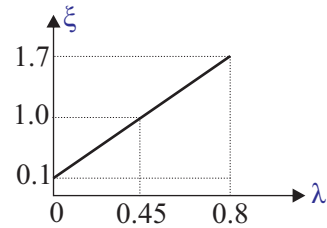
For $\lambda = 0.45$, characteristic equation will be $s^2 + 8s + 16 = 0$ or $(s + 4)^2 = 0$, both roots will be located at $s = -4$ and settling time $t_s = 4/4 = 1$ sec.

$$\text{For } \lambda = 0.075, \xi = 0.25, M_p = e^{-\frac{\pi \times 0.25}{\sqrt{1 - 0.25^2}}} = 0.44$$

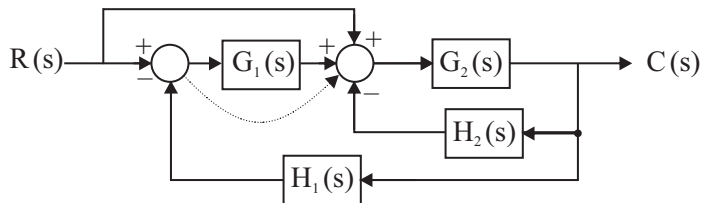
and peak unit step response, $y_{\max} = 1.44$

For $\lambda = 0, \xi = 0.1$ which is less than 0.25.

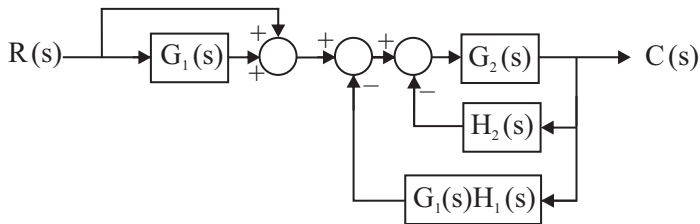
As ξ decreases, M_p increases and peak value of unit step response, will be more than 1.44.



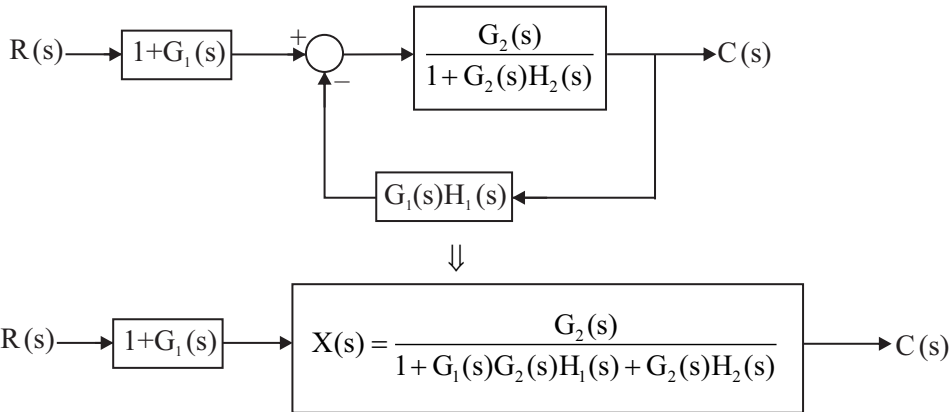
Q2(b)



⇓



⇓



Q3(a)
$$\frac{Y(s)}{R(s)} = \frac{100/s(s+2)}{1 + \frac{100(1+0.1s)}{s(s+2)}} = \frac{100}{s^2 + 12s + 100} \equiv \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$\omega_n = 10 \text{ rad/sec}, \xi = 0.6, \omega_d = \omega_n \sqrt{1 - \xi^2} = 8 \text{ rad/sec},$

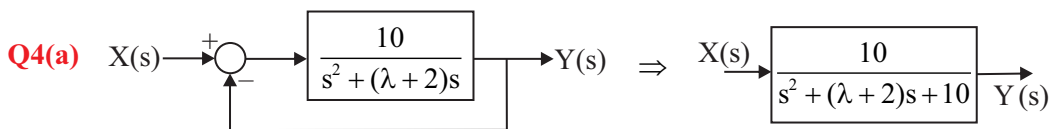
$\tan \phi = \frac{\sqrt{1 - \xi^2}}{\xi} = \frac{4}{3} \Rightarrow \phi = \tan^{-1}(4/3) = 53.13^\circ \text{ and } \sigma = \xi\omega_n = 6$

Note that the system has two complex conjugate poles and no zero.

For $r(t) = u(t), y(t) = -\frac{e^{-\sigma t}}{\sqrt{1 - \xi^2}} \sin(\omega_d t + \phi) = 1 - \frac{e^{-6t}}{\sqrt{1 - 0.6^2}} \sin(8t + 53.13^\circ)$
 $= 1 - 1.25e^{-6t} \sin(8t + 53.13^\circ)$

Compare with $y(t) = 1 - a e^{-bt} \sin(ct + d)$ to get

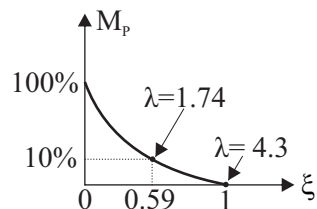
$a = 1.25, b = 6, c = 8 \text{ and } d = 53.13^\circ$



$\omega_n = \sqrt{10} \text{ and } \xi = \frac{\lambda + 2}{2\sqrt{10}}$

$e^{-\pi\xi/\sqrt{1-\xi^2}} \leq 0.1 \Rightarrow \xi \geq 0.59$

$\frac{\lambda + 2}{2\sqrt{10}} \geq 0.59 \Rightarrow \lambda \geq 1.74$



For $\lambda < 1.74$, peak overshoot will be larger than 10% and for $\lambda \geq 4.3$, system will not exhibit underdamped dynamics. In fact, $\xi \geq 1$ for $\lambda \geq 4.3$ and therefore, system will exhibit exponential dynamics.

$$\text{Q5(b)} \quad \frac{Y(s)}{R(s)} = \frac{10}{s^2 + 2s + 10} \Rightarrow \omega_n = \sqrt{10}, \xi = \frac{1}{\sqrt{10}}, \omega_d = \omega_n \sqrt{1 - \xi^2} = \sqrt{10} \times \sqrt{1 - \frac{1}{10}} = 3 \text{ rad/sec}$$

$$\phi = \tan^{-1} \frac{\sqrt{1 - \xi^2}}{\xi} = \tan^{-1} \left[\frac{\sqrt{1 - \frac{1}{10}}}{\frac{1}{\sqrt{10}}} \right] = 1.249 \text{ rad}$$

$$\text{Rise time, } t_1 = \frac{\pi - \phi}{\omega_d} = \frac{\pi - 1.249}{3} = 0.63 \text{ sec}$$

$$\text{Peak time, } t_2 = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.05 \text{ sec}$$

$$\% \text{ peak overshoot} = 100 e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} = 100 e^{-(\pi/\sqrt{10})/\sqrt{1-\frac{1}{10}}} = 35.09\%$$

$$\Delta = 35.09 \times \frac{12}{100} = 4.2$$

$$\text{Q6(c)} \quad \text{Closed loop transfer function, } T(s) = \frac{Y(s)}{X(s)} = \frac{ks + b}{s^2 + as + b}$$

$$\text{and open loop transfer function, } G(s) = \frac{ks + b}{s^2 + (a - k)s} = \frac{ks + b}{s(s + a - k)}$$

$$\text{Velocity error constant, } K_v = \lim_{s \rightarrow 0} s G(s) = \frac{b}{a - k} \quad \text{and} \quad e(\infty) = \frac{1}{K_v} = \frac{a - k}{b}$$

$$S_a^{e(\infty)} = \frac{\partial}{\partial a} e(\infty) \times \frac{a}{e(\infty)} = \frac{1}{b} \times \frac{a}{(a - k)} \times b = \frac{a}{a - k}$$

$$S_b^{e(\infty)} = \frac{\partial}{\partial b} e(\infty) \times \frac{b}{e(\infty)} = -(a - k) b^{-2} \times \frac{b}{a - k} \times b = -1$$

Q7. Routh array

s^5	1	6	25	\uparrow array segment for divisor polynomial \downarrow
s^4	1	6	25	
s^3	$\cancel{1}$	$\cancel{3}$		
s^2	3	25		
s^1	$-\frac{16}{3}$			
s^0	25			

auxiliary polynomial, $A(s) = s^4 + 6s^2 + 25$

$$\frac{d}{ds} A(s) = 4s^3 + 12s$$

Two changes in algebraic sign in array segment for divisor polynomial indicates that there are 2 RHP roots and so only there are 2LHP roots by virtual of symmetry .
All together , there are 3LHP and 2RHP roots. Thus, L=3, M=2, N=0 and $|L-M-N|=1$

Q8 a Characteristic polynomial, $P(s) = s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1$

s^6	1	3	3	1	
s^5	1	3	2		
s^4	0	1	1		← case of left column zero

Replace s by s^{-1} to get new polynomial, $Q(s)$ in s as written below.

$$Q(s) = s^{-6} + s^{-5} + 3s^{-4} + 3s^{-3} + 3s^{-2} + 2s^{-1} + 1 = 0 \quad \text{or} \quad Q(s) = s^6 + 2s^5 + 3s^4 + 3s^3 + 3s^2 + s + 1 = 0$$

Routh array

s^6	1	3	3	1	
s^5	2	3	1		
s^4	3/2	5/2	1		
s^3	-1/3	-1/3			
s^2	1	1			
s^1	0	2	0		← Premature termination
s^0	1				$A(s) = s^2 + 1 = 0 \Rightarrow s = \pm j1$

$$\frac{d}{ds} A(s) = 2$$

Two changes in algebraic sign in the left column above s^2 row, indicate 2 RHP roots. The premature termination indicates 2 IA roots at $s = \pm j1$. The remaining 2 roots are obviously in the LHP.

Q9(a) Characteristic equation is

$$1 + \frac{k e^{-sT}}{s(1+as)(1+bs)} = 0 \quad \text{or} \quad abs^3 + (a+b)s^2 + s + k^{-sT} = 0$$

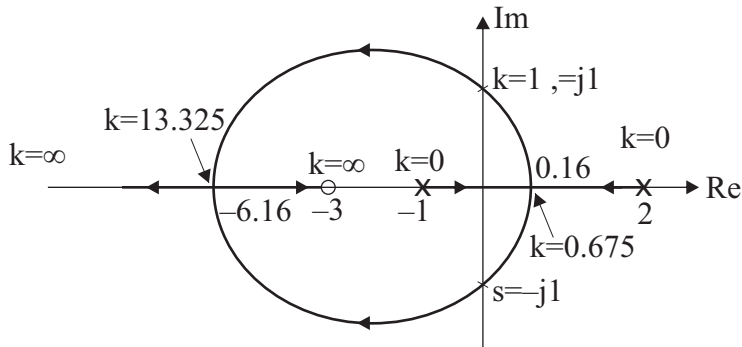
Use approximation $e^{-sT} = 1 - sT$ obtained by retaining first two terms of exponential series and truncating the rest, to get $ab s^3 + (a+b)s^2 + (1-kT)s + k = 0$

Routh array

s^3	ab	1-kT	
s^2	(a+b)	k	For the system to be stable $k > 0$.
s^1	$\frac{(a+b)(1-kT) - abk}{a+b}$		and $(a+b)(1-kT) - abk > 0$
s^0	k		or $k < \frac{a+b}{(a+b)T + ab}$

as $T \rightarrow 0$, $k < \frac{(a+b)}{ab}$

$$\mathbf{Q10(c)} \quad s^2 + (k-1)s + 3k - 2 = 0 \Rightarrow 1 + \frac{k(s+3)}{s^2 - s - 2} = 0 \text{ or } 1 + \frac{k(s+3)}{(s+1)(s-2)} = 0$$



The root locus for loop transmittance $G(s) = \frac{k(s+3)}{(s+1)(s-2)}$ is sketched and relevant values of k , are demonstrated there on.

$$\text{Break away/in points: } \frac{dk}{ds} = 0 \Rightarrow -\frac{d}{ds} \left[\frac{s^2 - s - 2}{s+3} \right] = 0 \Rightarrow s^2 + 6s - 1 = 0 \text{ and } s = 0.16, -6.16$$

IA crossing point: Characteristic equation is $s^2 + (k-1)s + 3k - 2 = 0$

Obviously, for $k=1$ the characteristic equation $s^2 + 1 = 0$ gives IA crossing points at $s = \pm j1$.

$$k|_{s=0.16} = \frac{1.16 \times 1.84}{3.16} = 0.675$$

$$k|_{s=-6.16} = \frac{5.16 \times 8.16}{3.16} = 13.325$$

Now, it is easy to conclude the following in regard to the system behaviour.

- (i) It is unstable for $k < 1$.
- (ii) It oscillates at 1 rad/sec for $k = 1$.
- (iii) It is stable for $k > 1$.
- (iv) For $1 < k < 13.325$, it exhibits underdamped dynamics.
- (v) For $k < 13.325$, it exhibits overdamped dynamics due to two distinct, negative and real roots.
- (vi) For $k = 13.325$, it does exhibit critically damped dynamics due to two negative, real and equal roots but for $k = 0.675$, it exhibits exponentially growing dynamics due to two real, equal but positive roots located at $s = 0.16$.

Q11. The root locus for $G(s) = \frac{K\left(s + \frac{2}{3}\right)}{s^2(s+6)}$; $0 \leq K \leq \infty$, is sketched below.

$$\sigma_A = \frac{-6 - \left(-\frac{2}{3}\right)}{2} = -\frac{8}{3}$$

$$\theta_A = \pm 90^\circ$$

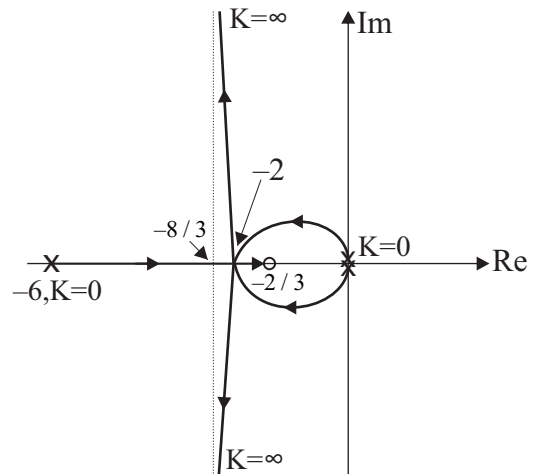
$$1 + \frac{K\left(s + \frac{2}{3}\right)}{s^2(s+6)} = 0 \Rightarrow K = \frac{-s^2(s+6)}{\left(s + \frac{2}{3}\right)}$$

Break away / in point :

$$\frac{dK}{ds} = 0 \Rightarrow -\frac{d}{ds} \frac{s^2(s+6)}{\left(s + \frac{2}{3}\right)} = 0$$

$$2s^3 + 8s^2 + 8s = 0 \text{ or } 2s(s^2 + 4s + 4) = 0 \text{ gives } s = 0, -2, -2.$$

$$K|_{s=-2} = \frac{4 \times 2 \times 2}{\left(\frac{4}{3}\right)} = 12$$



Q12(d). Use matrix equation to get

$$\dot{x}_1 = x_2 + a r(t) \text{ and } \dot{x}_2 = -2x_1 - 3x_2 + br(t); r(t) = u(t) \Rightarrow R(s) = \frac{1}{s}$$

$$sX_1(s) = X_2(s) + \frac{a}{s} \text{ and } sX_2(s) = -2X_1(s) - 3X_2(s) + \frac{b}{s}$$

$$X_1(s) = \frac{X_2(s)}{s} + \frac{a}{s^2} \text{ and } X_2(s) = -\frac{2}{s+3} X_1(s) + \frac{b}{s(s+3)}$$

$$X_2(s) = -\frac{2}{s+3} \left[\frac{X_2(s)}{s} + \frac{a}{s^2} \right] + \frac{b}{s(s+3)} \text{ gives } X_2(s) = -\frac{-2a + bs}{s(s+1)(s+2)}$$

Substitute $X_2(s) = -\frac{-2a + bs}{s(s+1)(s+2)}$ in the expression of $X_1(s)$ to get

$$X_1(s) = \frac{a}{s^2} + \frac{1}{s} \left[\frac{-2a + bs}{s(s+1)(s+2)} \right] = \frac{a(s+3) + b}{s(s+1)(s+2)}$$

$$x_1(\infty) = \lim_{s \rightarrow 0} sX_1(s) = \frac{1}{2}(3a + b) \text{ and } x_2(\infty) = \lim_{s \rightarrow 0} sX_2(s) = -a$$

$$-a = 1 \Rightarrow a = -1 \text{ and } \frac{1}{2}(3a + b) = \frac{1}{2}\{3 \times (-1) + b\} = 1 \Rightarrow b = 5$$

Alternatively, one may also solve by finding STM $\Phi(t)$ as follows.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \text{ and } \phi(s) = (sI - A)^{-1} = \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right]^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}$$

$$\phi(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)Bu(\tau)d\tau = \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \cdot 1d\tau ; u(\tau) = 1$$

$$= \int_0^t \begin{bmatrix} (2a+b)e^{-(t-\tau)} - (a+b)e^{-2(t-\tau)} \\ -(2a+b)e^{-(t-\tau)} + 2(a+b)e^{-2(t-\tau)} \end{bmatrix} d\tau = \begin{bmatrix} (2a+b)e^{-(t-\tau)} - \frac{1}{2}(a+b)e^{-2(t-\tau)} \\ -(2a+b)e^{-(t-\tau)} + (a+b)e^{-2(t-\tau)} \end{bmatrix} \Bigg|_0^t$$

$$x(t) = \begin{bmatrix} \frac{1}{2}(3a+b) - (2a+b)e^{-t} + \frac{1}{2}(a+b)e^{-2t} \\ -a + (2a+b)e^{-t} - (a+b)e^{-2t} \end{bmatrix} \text{ and } \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(3a+b) \\ -a \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{2}(3a+b) = 1 \text{ and } -a = 1 \Rightarrow a = -1 \text{ and } b = 5$$

Q13(c)

s^7	1	2	-1	-2	← $A(s) = s^6 + 2s^4 - s^2 - 2$
s^5	3	4	-1		$\frac{d}{ds} A(s) = 6s^5 + 8s^3 - 2s = 2[3s^5 + 4s^3 - 1]$
s^4	$\frac{2}{3} 1$	$-\frac{2}{3} -1$	-3		← entire row multiplied by 3/2
s^3	7	8			
s^2	$-\frac{15}{7} -15$	-21			← entire row multiplied by 7
s^1	$-\frac{27}{15} -4$				← entire row multiplied by 20/9
s^0	-21				

Scan the left column from top to bottom to observe one change in algebraic sign. This indicates one RHP root and so only another in LHP due to symmetry.

$$A(s) = s^4(s^2+2) - (s^2+2) = (s^4-1)(s^2+2) = (s^2+1)(s+1)(s-1)(s^2+2) = 0 \text{ gives } s = \pm 1, \pm j1, \pm j\sqrt{2}$$

Thus, there are 4 roots on $j\omega$ axis, 1 RHP root and 2LHP roots, that is, $p = 2$, $q = 1$ and $r = 4$

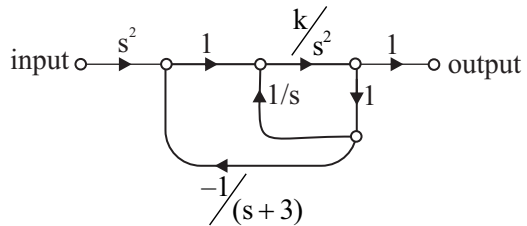
Q14. Use equivalent signal flow graph as sketched below , to get the transfer function

$$G(s) = \frac{k}{1 + \frac{1}{s(s+3)} + \frac{k}{s^2(s+3)}} = \frac{k s^2(s+3)}{s^2(s+3) + s + k}$$

The characteristic equation is $s^3 + 3s^2 + s + k = 0$

Routh array

s^3	1	1
s^2	3	k
s^1	$\frac{3-k}{3}$	
s^0	k	



For $k=3$, s^1 row has all zero entries and divisor polynomial $A(s) = 3s^2 + k = 3s^2 + 3 = 0$ gives $s = \pm j1$ that is, there will be 2 IA roots .The third root will obviously be in the LHP. Thus , $k=3$.

Q15(b) Construct Routh array for $D(s) = s^4 + k_1 s^3 + s^2 + k_2 s + 1$ as follows.

s^4	1	1	1
s^3	k_1	k_2	
s^2	$\frac{k_1 - k_2}{k_1}$	1	
s^1	$\frac{k_1 k_2 - (k_2^2 + k_1^2)}{k_1 - k_2}$		
s^0	$\frac{k_1 - k_2}{1}$		

For 2 IA roots s^1 row must have all zero entries , that is ,

$$\frac{k_1 k_2 - (k_2^2 + k_1^2)}{k_1 - k_2} = 0 \text{ or } k_1^2 + k_2^2 = k_1 k_2$$

Q16(b) Position error constant , $K_p = \lim_{s \rightarrow 0} \frac{k(s+7)}{s^2 + 2s + 10} = \frac{7k}{10}$

$$e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 + \frac{7k}{10}} = \frac{10}{7k + 10}$$

$$S_k^{e(\infty)} = \frac{\partial}{\partial k} e(\infty) \times \frac{k}{e(\infty)} = \frac{-70}{(7k + 10)^2} \times \frac{k}{10} \times (7k + 10) = \frac{-7k}{7k + 10}$$

$$|S_k^{e(\infty)}|_{k=10} = \frac{7 \times 10}{7 \times 10 + 10} = \frac{7}{8}$$

Q17(d) The characteristic equations is $1 + \frac{k(s^2 - 2s + 4)}{s^2 + 2s + 2} = 0$ or $(k+1)s^2 + 2(1-k)s + 2 + 4k = 0$

The stability demands

$$(k+1) > 0 \quad \text{or} \quad k > -1$$

$$2(1-k) > 0 \quad \text{or} \quad k < 1$$

$$(2+4k) > 0 \quad \text{or} \quad k > -1/2$$

All together, stability range is $-\frac{1}{2} < k < 1$.

Q18. $1 + \frac{k(s+2)}{s^2 - 4s + 13} = 0$ gives $k = -\frac{s^2 - 4s + 13}{s + 2}$

$$\frac{dk}{ds} = 0 \quad \text{gives} \quad \frac{(2s-4)(s+2) - (s^2 - 4s + 13)}{(s+2)^2} = 0 \quad \text{or} \quad s^2 + 4s - 21 = 0 \quad \text{or} \quad s = -2 \pm 5$$

The point $s = 3$ does not lie on real axis segment of root locus.

Thus, the break away point lies at $s = -7$

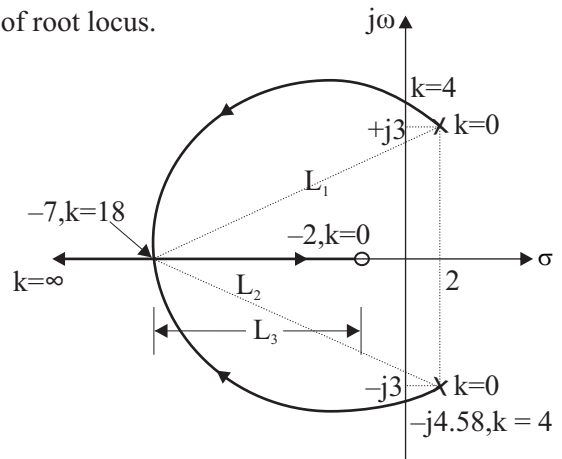
$$L_1 = L_2 = \sqrt{3^2 + 9^2} = 9.5 \quad \text{and} \quad L_3 = 5$$

$$k|_{s=-7} = \frac{L_1 L_2}{L_3} = \frac{9.5 \times 9.5}{5} = 18$$

The characteristic equation

$$s^2 + (k-4)s + 2k + 13 = 0 \quad \text{gives}$$

IA crossing points at $s = \pm j4.58$ for $k=4$.

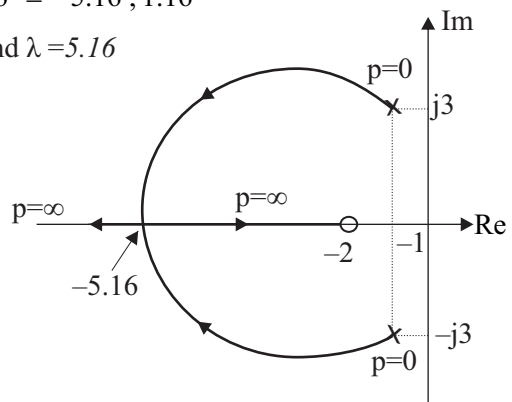


Q19 The characteristic equation is

$$1 + \frac{10}{(s+2)(s+p)} = 0 \quad \text{or} \quad s^2 + ps + 2s + 2p + 10 = 0 \quad \text{or} \quad 1 + \frac{p(s+2)}{s^2 + 2s + 10} = 0 \Rightarrow p = -\frac{s^2 + 2s + 10}{s + 2}$$

$$-\frac{dp}{ds} = 0 \quad \text{gives} \quad s^2 + 4s - 6 = 0 \quad \text{and} \quad s = -2 \pm 3.16 = -5.16, 1.16$$

The feasible break away point lies at $s = -5.16$ and $\lambda = 5.16$



Q20. $G(j\omega) = \frac{10}{j\omega(j\omega+1)}$. Let gain crossover frequency be ω_{gc}

$$|G(j\omega_{gc})| = 1 \text{ gives } \frac{10}{\omega_{gc}\sqrt{1+\omega_{gc}^2}} = 1 \Rightarrow \omega_{gc}^4 + \omega_{gc}^2 - 100 = 0$$

Solve the equation to get $\omega_{gc}^2 = 9.5$ or -10.05

Discard negative value so that $\omega_{gc} = \sqrt{9.5} = 3.08 \text{ rad/sec}$

$$\phi_{m1} = 180^\circ + \angle G(j\omega_{gc}), \quad \phi_{m2} = 180^\circ + \angle G(j\omega_{gc}) - 0.1\omega_{gc}$$

$$\phi_{m1} - \phi_{m2} = 0.1\omega_{gc} = 0.1 \times 3.08 = 0.308 \text{ rad} = 0.308 \times \frac{180^\circ}{\pi} = 17.65^\circ$$

Q21. The characteristic equation is $|sI - A| = 0$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ -K_1 & -(K_2+1) \end{bmatrix} = \begin{bmatrix} s+2 & -1 \\ K_1 & s+K_2+1 \end{bmatrix}$$

$$|sI - A| = 0 \Rightarrow (s+2)(s+K_2+1) + K_1 = 0 \text{ or } s^2 + (K_2+3)s + K_1 + 2K_2 + 2 = 0$$

$$\text{Settling time, } t_s = 0.5 \Rightarrow \frac{4}{\xi\omega_n} = 0.5 \text{ or } \xi\omega_n = 8$$

$$\text{Since, } \xi = 0.5, \omega_n = 16$$

$$2\xi\omega_n = K_2 + 3 = 16 \Rightarrow K_2 = 13$$

$$\omega_n^2 = (K_1 + 2K_2 + 2) = 16^2 \Rightarrow K_1 = 16^2 - 28 = 228$$

$$K_1 + K_2 = 13 + 228 = 241$$

Q22(b) $\dot{x}_1 = -5x_1 + x_2 + b_1u$

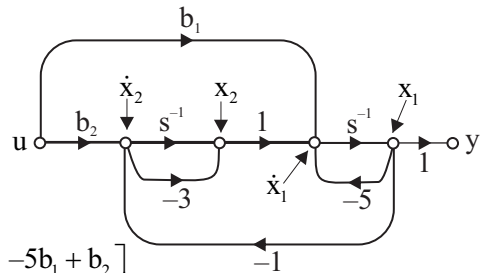
$$\dot{x}_2 = -x_1 - 3x_2 + b_2u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$\text{Controllability matrix, } Q_2 = [B : AB] = \begin{bmatrix} b_1 & -5b_1 + b_2 \\ b_2 & -b_1 - 3b_2 \end{bmatrix}$$

Since, the model is UNCONTROLLABLE, $|Q_c| = 0$, that is,

$$-b_1(b_1 + 3b_2) - b_2(-5b_1 + b_2) = 0 \Rightarrow (b_1 - b_2)^2 = 0 \Rightarrow b_1 = b_2$$



Q23. $sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$ and $\phi(s) = (sI - A)^{-1} = \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$

STM, $\phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \phi(t)x(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+t \\ 1 \end{bmatrix}$

$$\left[x_1^2(t) + x_2^2 \right]_{t=1} = \left[(1+t)^2 + 1^2 \right]_{t=1} = 5$$

Alternate method

$$\dot{x}_1(t) = x_2(t) \text{ and } \dot{x}_2(t) = 0 \Rightarrow s X_1(s) - X_1(0) = X_2(s) \text{ and } s X_2(s) - x_2(0) = 0$$

$$X_2(s) = \frac{x_2(0)}{s} = \frac{1}{s} \Rightarrow x_2(t) = 1 \Rightarrow x_2(1) = 1$$

$$X_1(s) = \frac{X_2(s)}{s} + \frac{x_1(0)}{s} = \frac{1}{s^2} + \frac{1}{s} \Rightarrow x_1(t) = t+1 \Rightarrow x_1(1) = 2$$

$$x_1^2(1) + x_2^2(1) = 2^2 + 1^2 = 5$$

Q24. $\dot{x}_1 = x_1 + \frac{1}{2}x_2 + r$

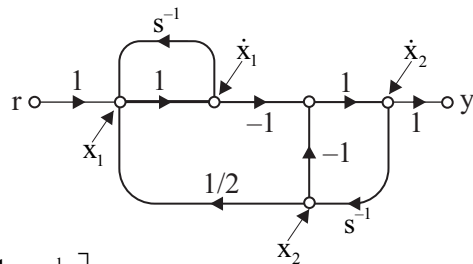
$$\dot{x}_2 = -\dot{x}_1 - x_2$$

$$= -x_1 - \frac{1}{2}x_2 - r - x_2$$

$$= -x_1 - \frac{3}{2}x_2 - r$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} r \Rightarrow A = \begin{bmatrix} 1 & \frac{1}{2} \\ -1 & -\frac{3}{2} \end{bmatrix}$$

$$\lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22} = 1 + \frac{1}{2} - 1 - \frac{3}{2} = -1$$



Q25. $G_1(s) = 10$, $G_2(s) = \frac{1}{s\left(1 + \frac{s}{0.6}\right)} = \frac{0.6}{s(s+0.6)}$; hint: $20 \log \frac{K}{\omega} \Big|_{\omega=1} = 0 \Rightarrow K = 1$

$$G_3(s) = 3s; \text{ hint: } 20 \log (3\omega) \Big|_{\omega=1} = 9.54$$

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s) + G_2(s)G_3(s)} = \frac{6}{s^2 + 2.4s + 6}$$

$$\omega_n = \sqrt{6} \text{ and } \xi = \frac{2.4}{2\sqrt{6}} = 0.48989 \cong 0.5$$

